MA4235 Topics in Graph Theory

- Graph notions:
  - Identical: $V(G) = V(H)$ and $E(G) = E(H)$.
  - Isomorphic: $\exists$ bijective mapping f from $V(G)$ to $V(H)$ s.t. $uv \in E(G) \iff f(u)f(v) \in E(H)$
  - H is a subgraph of G : $V' \subseteq V$ and $E' \subseteq E$ and all endpoints of edges in E' are in V'.
    $H = (V', E')$, $G = (V, E)$
  - Induced subgraph: subgraph of G induced on $S \subseteq V(G)$: $G[S] = (S, \{uv : u, v \in S, uv \in E(G)\})$
  - Neighbour set: $N_G(v) \subseteq V(G)$
    - Neighbour set of a set of vertices: $N(U) := \{v : v \in V \setminus U,\ vu \in E(G) \text{ for some } u \in U\}$, $U \subseteq V$
  - Degree: $d_G(v) = |N_G(v)|$
    - $\delta(G) := \min_{v \in V(G)} d(v)$
    - $\Delta(G) := \max_{v \in V(G)} d(v)$
    - $d(G) :=$ average degree of $G = \dfrac{2|E(G)|}{|V(G)|}$
  - Theorems:
    - number of vertices with odd degree is even
    - given a non-empty G, it has a subgraph $H \subseteq G$ s.t. $\delta(H) \geq \frac{1}{2} \cdot d(G)$
      - PF: repeatedly remove vertices with degree $< \frac{1}{2} d(G)$. We can show that there must still be edges remaining.
- Path: no repeated vertices, and start is distinct from end.
- Cycle: start and end is the same
- Walk: can repeat vertices, but not edges
  - every walk from x to y contains a path from x to y.
  - Closed walk: walk that start and end at same vertex.
- Girth: $g(G) :=$ min cycle length
- Circumference: max cycle length
- Thm: · Every graph G has a path of length $\delta(G)$
  · Every graph G contains a cycle of length $\geq \delta(G) + 1$ if $\delta(G) \geq 2$
  } consider the longest path. the end vertex must only be adjacent to other vertices on the path.
- Distance: $d_G(x, y) :=$ length of shortest path connecting x and y in G
- Diameter: max distance over all pairs of vertices
- Thm: $g(G) \leq 2 \cdot \text{diam}(G) + 1$
- Adjacency matrix: A_G : $|V| \times |V|$ matrix with $a_{ij} = \begin{cases} 1 & \text{if } i \text{ is adj. to } j \\ 0 & \text{otherwise} \end{cases}$
  - can use to count the number of walks/closed walks of a certain length in G
- Connected: any two of its vertices have a path connecting them
  - The vertices of a connected graph G can be enumerated s.t. $\underbrace{G[v_1, \dots, v_i]}_{\text{induced subgraph with vertices } v_1, \dots, v_i}$ is connected for all $i = 1, \dots, n$
- Connected components: maximal connected subgraphs
- X separates A and B : every path from every $a \in A$ and $b \in B$ contains a vertex or edge from X.
  $X \subseteq V(G) \cup E(G)$, $A \subseteq V(G)$, $B \subseteq V(G)$
- Cut vertex (or edge): a vertex that causes a graph to become disconnected when removed
- k-vertex-connected: removing (strictly) less than k vertices does not disconnect the graph.
  - Thm: every 2-vertex-connected graph contains a cycle.
  - $K(G)$: max k s.t. G is k-vertex-connected (For complete graphs, $K(G) = n-1$)
  - Thm: $K(G) \leq \delta(G)$
  - Thm: every graph of average degree $\geq 4k$ has a k-connected subgraph (PF: by induction on subgraphs)
- Dirac's thm (extended): If G is connected and $\delta(G) \geq \frac{k}{2}$ then G contains a path of length $\min\{2\delta(G), v(G) - 1\}$.

- **Forest**: graph with no cycle
- **Tree**: connected forest
- **Equiv statements**:
  - T is a tree
  - any two vertices of T are connected by a unique path
  - T is minimally connected (i.e. removing any edge disconnects the graph)
  - T is maximally cycle-free (i.e. adding any edge creates a cycle)
- **Tree thms**:
  - the vertices of a tree can be ordered such that the i^{th} vertex has a unique neighbour in $\{v_1, \dots, v_{i-1}\}$
  - T is a tree $\Leftrightarrow$ T is connected and has $n-1$ edges
- G is **k-colourable**: can use k colours to colour its vertices s.t. every pair of adj. vertices receive different colours
- **Chromatic number**: $\chi(G)$: min k s.t. G is k-colourable.
- **Bipartite graph**: $\chi(G) \leq 2$
- **r-partite graph**: $\chi(G) \leq r$
- **Thm**: G is bipartite $\Leftrightarrow$ G has no odd cycles

  - $\chi(C_n) = 2$ if n is even
  - $\chi(C_n) = 3$ if n is odd
  - $\chi(K_n) = n$

- **Matching**: set of vertex-disjoint edges
- $\nu(G)$: max matching size
- **Perfect matching**: $\nu(G) = \frac{|V(G)|}{2}$
- In a bipartite matching:
  - **Alternating path**: path starting with unmatched edge and alternating between unmatched and matched thereafter.
  - **Augmenting path**: alternating path that ends with an unmatched edge.
  - **Thm**: matching is optimal $\Leftrightarrow$ no augmenting paths
- **Vertex cover**: set of vertices s.t. every edge is incident to some vertex in this set.
- **vertex covering number**: $\tau(G) :=$ num. of vertices in a min vertex cover
- **Thm**: $\tau(G) \geq \nu(G)$
- **König's thm**: In any bipartite graph, $\tau(G) = \nu(G)$
- **LP duality** of max matching & min vertex cover: $\nu(G) \leq \underbrace{\nu^*(G) = \tau^*(G)}_{\text{these two problems are LP duals}} \leq \tau(G)$

  $\nu^*(G)$: (non-integer) LP version of max matching

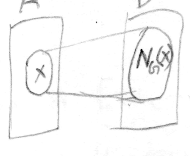
  $\tau^*(G)$: (non-integer) LP version of min vertex cover

- For any graph G:
  - $\tau(G) + \alpha(G) = n$

    $\tau(G)$: MVC, $\alpha(G)$: MIS

- **Hall's thm**: Given G bipartite on $A \sqcup B$:

  G has a matching perfect to A $\Leftrightarrow \forall X \subseteq A,\ |N_G(X)| \geq |X|$



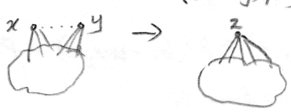
- **Flow network**: directed graph with special vertices s (source) and t (sink), each edge has a capacity c_e
  - **Flow**: $f: E(G) \to \mathbb{R}_{\geq 0}$ s.t.

    $$\begin{cases} \cdot\ 0 \leq f(e) \leq c_e \quad \forall e \in E(G) \\ \cdot\ \sum_{u \to v} f(u \to v) = \sum_{v \to w} f(v \to w) \quad \forall v \in V(G) \setminus \{s,t\} \quad \text{(conservation of flow)} \end{cases}$$

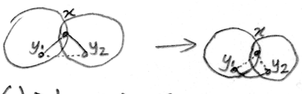
  - **Cut**: partition of $V(G)$ into $S \sqcup T$ where $s \in S$ and $t \in T$
    - **capacity** of cut: $\sum_{\substack{u \in S \\ v \in T \\ u \to v}} c_{u \to v}$ (i.e. only edges from S to T are considered, but not those from T to S)
  - **Max-flow min-cut thm**: max size of flow = min size of cut
  - **Integral flow thm**: if all capacities are integers then max flow must be attainable with integral flows
- **Thm**: any d-regular bipartite graph can be decomposed into edge-disjoint perfect matchings. (Proof by induction and Hall's)

- **Tutte's thm:** $G=(V,E)$ has a perfect matching $\iff \forall U \subseteq V$, $G[V-U]$ has at most $|U|$ connected components with an odd number of vertices
  - (G: not necessarily bipartite)
- **Deficiency** of a set $X \subseteq A$: $\mathrm{def}(X) = |X| - |N_G(X)|$
  - where $G = A \sqcup B$ is bipartite
- **Extended Hall's thm:** If G is bipartite on $A \sqcup B$, then $\nu(G) = |A| - \max_{X \subseteq A} \mathrm{def}(X)$
  - $\nu(G)$: max matching size
- **Tutte-Berge formula** (Extended Tutte's thm): Given a general graph G, $\nu(G) = \frac{1}{2} \min_{U \subseteq V(G)} \left(|U| - o(G[V(G)-U]) + |V(G)| \right)$
  - o: # of odd components
- **Thm:** Every 3-regular graph with no cut edge has a perfect matching
- **Hypergraph:** an edge can have any number of vertices
  - **k-uniform hypergraph:** each edge has k vertices
  - $\nu(H)$: max number of vertex-disjoint edges in H (hypergraph)
  - **Thm:** In an r-uniform r-partite hypergraph H, $\nu(H) \leq \tau(H) \leq r \cdot \nu(H)$
    - $\nu(H)$: max matching size; $\tau(H)$: min vertex cover size
- **Path cover:** Given a directed graph, a path cover is a set of vertex-disjoint paths covering all vertices.
- **Thm** (Gallai-Milgram): Every directed graph D has a path cover of at most $\alpha(D)$ paths.
  - α: MIS
  - $\mathrm{ter}(P)$: terminating vertex of path P.
- **Partially ordered set** $(P, \leq)$: P is a set, $\leq$ is a binary relation over P satisfying reflexivity, antisymmetry, transitivity
  - reflexivity: $\forall a \in P: a \leq a$
  - antisymmetry: $\forall a,b \in P: a \leq b$ and $b \leq a \Rightarrow a = b$
  - transitivity: $\forall a,b,c \in P: a \leq b$ and $b \leq c \Rightarrow a \leq c$
- **Totally ordered set** $(P, \leq)$: partially ordered set where every pair of elements are comparable
  - $\forall a,b \in P: a \leq b$ or $b \leq a$
- **Dilworth's thm:** Given a finite poset P,
  min # of chains covering P = max # of elements in an antichain
  - antichain: a set in which every 2 elements are incomparable
- **Chromatic number:** $\chi(G)$ = min colours needed to colour G.
- **Four-colour thm:** G is planar $\Rightarrow \chi(G) \leq 4$
- **Grötzsch thm:** G is planar and does not contain $K_3 \Rightarrow \chi(G) \leq 3$
- **Prop:** For any graph G, $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$
  - m: #edges
  - For a tight example, consider $G = K_n$.
- **Prop:** $\chi(G) \leq \Delta(G) + 1$
  - **Pf:** greedy colouring
  - **Thm:** If G is an odd cycle or is complete, then $\chi(G) = \Delta(G) + 1$. Otherwise, $\chi(G) \leq \Delta(G)$.
- **Prop tighter greedy colouring:** $\chi(G) \leq \mathrm{col}(G) = \max_{\text{induced } H \subseteq G} \delta(H) + 1$
  - $\mathrm{col}(G)$: the least k s.t. G has an ordering in which every vertex is preceded by fewer than k of its neighbours
- **k-constructible graph:**
  1. K_k is k-constructible
  2. If G is k-constructible and x is not adjacent to y in G then $(G+xy)/xy$ is k-constructible

     G → $(G+xy)/xy$

  3. If G_1 and G_2 are both k-constructible and $V(G_1) \cap V(G_2) = \{x\}$, and $\exists y_i \in V(G_i)$ s.t. $xy_i \in E(G_i)$, then
     $G = (G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$
     is k-constructible
     - Hajós join

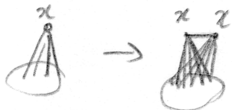
Thm (Hajós): $\chi(G) \geq k \iff G$ has a k-constructible subgraph

- k-critical: $\chi(G)=k$ and removing any edge or vertex will decrease the chromatic number (by 1)
- Prop: G is k-critical $\Rightarrow$ G is k-constructible
  - Pf: any prober subgraph of G cannot be k-constructible.
- Edge colouring: colouring of edges s.t. no two edges sharing a vertex have the same colour
- Edge chromatic number: $\chi'(G)$: min number of colours needed for an edge-colouring
- Prop: $\chi'(G) \geq \Delta(G)$
- Prop: If G is bipartite & d-regular: $\chi'(G)=\Delta(G)=d$
- Thm (König): If G is bipartite: $\chi'(G)=\Delta(G)=\chi(L(G))$
- Thm (Vizing): Every (simple) graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$
  - Type I graphs: $\chi'(G)=\Delta(G)$ (includes all bipartite, even cycles, even complete)
  - Type II graphs: $\chi'(G)=\Delta(G)+1$ (includes all add cycles, odd complete)
- Thm (Extended Vising for multigraphs): $\Delta(G) \leq \chi'(G) \leq \Delta(G)+\mu(G)$

  $\mu(G)$: max multiplicity of edges
- List chromatic number / Choice number: $ch(G)$:

  min k s.t. if each vertex has k colours to choose from, we can pick a colour for each vertex so that no two adjacent vertices have the same colour.
- Cor: $ch(G) \geq \chi(G)$
- Thm (on boundedness of avg. degree w.r.t. $ch(G)$) (Alan): There exists a function $f: \mathbb{N} \to \mathbb{N}$ s.t. for any k, all graphs with avg. degree $> f(k)$ satisfies $ch(G) \geq k$.
- Thm (Thomassen): Every planar graph has $ch(G) \leq 5$.
- List edge colouring number: $ch'(G)$
- Thm (Galvin): $ch'(G)=\chi'(G)$
- A subset $U \subseteq V(D)$ (D: directed graph) is a kernel: $\forall v \in V(D) \setminus U$, there is an edge from v to a vertex in U.

- Clique number: $\omega(G)$: largest clique size in G.
  - $\chi(G) \geq \omega(G)$
- G is perfect if for every subgraph $H \subseteq G$, $\chi(H)=\omega(H)$
  - E.g. all complete graphs are perfect
  - Prop: The complement of any bipartite graph is perfect.
  - If G is bipartite then $L(G)$ is perfect.
  - Thm (Lovász) (weak perfect graph thm): G is perfect $\Leftrightarrow$ $\overline{G}$ is perfect
  - "Expanding" a vertex:



  - Lemma: Expanding any vertex of a perfect graph yields a perfect graph.
- Chromatic polynomial: $P(G,k)$:= number of proper k-colourings of G
  - it is polynomial in k
  - E.g. $P(K_3,k)=k(k-1)(k-2)$
  - E.g. $P(K_t,k)=k(k-1)\cdots(k-t+1)$
  - E.g. $P(P_n,k)=k(k-1)^n$ (P_n: path of length n)
  - E.g. $P(C_n,k)=(k-1)^n+(k-1)(-1)^n$
  - Thm deletion-contraction: $P(G,x)=P(G-uv,x)-P(G/uv,x)$ $\forall uv \in E(G)$
    - $G-uv$: remove uv from G
    - G/uv: contract uv (i.e. merge vertices u and v)

- Chromatic polynomial thms: Given G with n vertices and m edges:
  - $\deg(P(G,x)) = n$
  - The leading term of $P(G,x)$ is x^n and the second term is $-mx^{n-1}$
  - The constant term of $P(G,x)$ is 0
  - If G is connected then the coefficients of $x^n, \dots, x^1$ are all nonzero and alternate in signs
- Four-colour theorem: If G is planar then $P(G,4) > 0$
- The absolute values of coefs. of $P(G,x)$ is log-concave (and hence unimodal), i.e. $a_i^2 \geq a_{i-1}a_{i+1}$
- Orientation of a graph: a choice of direction for each edge
  - Acyclic: no directed cycle
  - Thm number of acyclic orientations: $a(G) = (-1)^n P(G,-1)$
- Ramsey number: $R(k,\ell) :=$ min t s.t. every red/blue edge-colouring of K_t has either a red K_k or blue K_ℓ or both.
  - E.g. $R(3,3) = 6$ (in K_6 there must be a red or blue K_3)
  - $R(k) := R(k,k)$
  - Fact: $R(k,\ell) = R(\ell,k)$
  - E.g. $R(2,2) = 2$
  - E.g. $R(4,4) = 18$
  - Paley graphs (to lower-bound $R(k)$): Given a prime p s.t. $p \equiv 1 \pmod 4$, $V(G) := \mathbb{Z}/p\mathbb{Z}$, $E(G) := \{(x,y) : x - y \equiv a^2 \pmod p\}$ for some $a \not\equiv 0 \pmod p$

Ramsey theorem: for any k, ℓ: $R(k,\ell)$ exists.

- Thm: $R(k,k) \leq \binom{2k-2}{k-1} \sim \frac{4^k}{\sqrt{k}}$ $\quad (x \sim y := \frac{x}{y} \to 1)$
- Thm: $R(k,\ell) \leq \binom{k+\ell-2}{k-1}$
- Fact: $R(k,\ell) \leq R(k-1,\ell) + R(k,\ell-1)$
- Thm: $R(k,k) \in O\left(\frac{4^k}{k}\right)$
- Thm (Thomassen): $R(k,k) \leq 4R(k,k-2) + 2$
- Goodman's bound: In any red/blue edge-colouring of K_n, there are at least $\frac{1}{4}\binom{n}{3} + O(n^2)$ monochromatic K_3.
- Thm: $R(k,k) \geq \frac{k}{\sqrt{2}e}(\sqrt{2})^k$
- Thm: $R(3,k) \in \Theta\left(\frac{k^2}{\log k}\right)$
- k-uniform hypergraph: each edge contains k vertices
- K_n^k: complete k-uniform hypergraph on n vertices (i.e. every k-element subset of vertices has an edge)
- $R_k(s,t)$: min n s.t. every red/blue edge-colouring of K_n^k has either a red K_s^k or blue K_t^k or both
- Thm: $2^{ct^2} \leq R_3(t,t) \leq 2^{2^{4t}}$
- Thm: $2^{2^{c't^2}} \leq R_4(t,t) \leq 2^{2^{2^{ct}}}$
- Ramsey finiteness thm: $R_k(n_1, \dots, n_t)$ is finite
- Thm Erdös-Szekeres: $\forall m \geq 4$, $\exists n$ s.t. for any n points in $\mathbb{R}^2$ where no three are collinear, at least m of them are on their convex hull.
- Lemma: Given m points, if any four of them form a convex quadrilateral, then all m points are on their convex hull.
- Ramsey number for arbitrary graph: $R(G,H) :=$ min n s.t. every red/blue edge-colouring of K_n has either a red G or blue H or both.
- Thm for trees & complete graphs: Given any tree T with t vertices, $R(T,K_s) = (s-1)(t-1)+1$
- Thm for cycles: $R(C_t, C_t) = \begin{cases} \frac{3}{2}t - 1 & \text{if } t \geq 6 \text{ and } t \text{ is even} \\ 2t - 1 & \text{if } t \geq 5 \text{ and } t \text{ is odd} \end{cases}$
- Thm: $2^q < R(\underbrace{C_3, \dots, C_3}_{q \text{ times}}) \leq 3q!$ (i.e. grows at least exponentially fast)
- Thm: $R(\underbrace{C_4, \dots, C_4}_{q \text{ times}}) \leq q^2 + q + 1$ (i.e. grows at most polynomially fast)

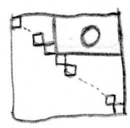
- Fact: $R(k_1, \dots, k_t) \leq R(k_1 - 1, \dots, k_t) + R(k_1, k_2 - 1, \dots, k_t) + \dots + R(k_1, \dots, k_t - 1)$
- If $R(s-1,t)$ and $R(s,t-1)$ are both even, then $R(s,t) \leq R(s-1,t) + R(s,t-1) - 1$
- Arithmetic Ramsey theory:
  - Van der Waerden's thm: Given $r, t \in \mathbb{N}$: $\exists N = N(r,t)$ s.t. $\forall n \geq N$, any colouring $c: \{1, \dots, n\} \to \{1, \dots, r\}$ must contain a t-term monochromatic arithmetic progression $a, a+d, \dots, a+(t-1)d$ $(x - 2y + z = 0)$
    - $W(r,t) :=$ min N s.t. Van der Waerden's thm holds
    - E.g. $W(2,3) \geq 9$
  - Szemerédi's thm: Given $A \subseteq \mathbb{N}$: If $\limsup_{n \to \infty} \frac{|A \cap \{1, \dots, n\}|}{n} > 0$, then for all $k \in \mathbb{N}$, A contains infinitely many arithmetic progressions of length k.
  - Primes thm: There are arbitrarily long (but finite) arithmetic progressions in the set of primes
  - $[t]^n := \{1, \dots, t\}^n$
  - Combinatorial line: $L := \{x \in [t]^n : x_i = a_i \text{ for } i \notin I,\ x_i = c \text{ for } i \in I,\ c \in [t]\}$ where $I \subseteq [n]$ and $a_i \in [t]$ for $i \notin I$.
    (so its kinda a constant on all the $i \notin I$ dimensions and a line on all the $i \in I$ dimensions)
  - Thm Hales, Jewett: Given $r, t \in \mathbb{N}$: $\exists n_0$ s.t. $\forall n \geq n_0$, any colouring $c: [t^n] \to \{1, \dots, r\}$ must contain a monochromatic combinatorial line.
  - Schur's thm: Given $k \in \mathbb{N}$: $\exists S = S(k)$ s.t. $\forall n \geq S$, any k-colouring of $\{1, \dots, n\}$ has a monochromatic solution of $x + y = z$ $(x + y - z = 0)$
  - Thm: $\forall m \geq 1$, $\exists p_0$ s.t. $\forall$ prime $p \geq p_0$, $x^m + y^m \equiv z^m \pmod{p}$ has solution s.t. $p \nmid xyz$.
  - Rado's thm (special case): Given a linear equation $\sum_{i=1}^{n} a_i x_i = 0$:
    $\exists$ nonempty $I \subseteq [n]$ s.t. $\sum_{i \in I} a_i = 0$ $\iff$ $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$, any colouring $c: [n] \to [r]$ must a monochromatic solution to $\sum_{i=1}^{n} a_i x_i = 0$
    (i.e. all $x_i \in [n]$ have the same colour)
- Given a graph H and $n \in \mathbb{N}$, what is the maximum number of edges of a H-free graph on n vertices? $(ex(n,H))$
  - $H =$  : max $e(G) = 0$
  - $H =$  : max $e(G) = \lfloor \frac{n}{2} \rfloor$
  - $H =$  : max $e(G) = \lfloor \frac{n^2}{4} \rfloor$ (complete bipartite graph) (Mantel thm)
  - Mantel ext:
    - If G has n vtxs & m edges, then G has at least $\frac{4m}{3n}(m - \frac{n^2}{4})$ triangles.
    - In any n vtx G, one can partition the edges into at most $\lfloor \frac{n^2}{4} \rfloor$ edges or triangles.
- Cauchy-Schwarz ineq.: $(\sum a_i^2)(\sum b_i^2) \geq (\sum a_i b_i)^2$
- $T_k(n)$: complete k-partite graph on n vertices whose partitions are "as equal as possible"
- $t_k(n)$: $e(T_k(n))$
- Turán's thm: $H = K_r$: max $e(G) = t_{r-1}(n)$. Furthermore, the only graph attaining this bound is $T_{r-1}(n)$.
  - Cor (Erdös): If S is a set of n points on the plane s.t. $\text{diam}(S) \leq 1$, then the number of pairs of points with distance $> \frac{1}{\sqrt{2}}$ is at most $t_3(n) = e(K_{\frac{n}{3}, \frac{n}{3}, \frac{n}{3}}) = \frac{n^2}{3}$.
- Turán's number: $ex(n,H) := \max e(G)$ where G is H-free and has n vertices.
- Turán density: $\pi(H) := \lim_{n \to \infty} \frac{ex(n,H)}{\binom{n}{2}}$ (= proportion of edges) (note: $\frac{ex(n,H)}{\binom{n}{2}}$ is non-increasing as $n \to \infty$)
- Thm: $\pi(H) = 1 - \frac{1}{\chi(H) - 1}$ $\to$ Cor: H is bipartite $\iff \pi(H) = 0$
- Thm: For any $r \geq 2$, $s \geq 1$, $\varepsilon > 0$: $\exists N \in \mathbb{N}$ s.t. $\forall G$ with $n \geq N$ vertices and $e(G) \geq (1 - \frac{1}{r-1} + \varepsilon)\binom{n}{2}$, $G \supseteq K_{\underbrace{s, \dots, s}_{r}}$ (Note: $K_{\underbrace{s, \dots, s}_{r}} \supseteq K_r$)
- Goodman's bound: The number of monochromatic K_3 in any red/blue colouring of K_n is $\geq (\frac{1}{4} + o(1))\binom{n}{3}$
- Thm: For any $n \geq 4$: $ex(n, K_{2,2}) \leq \frac{n}{4}(1 + \sqrt{4n-3}) \sim \frac{1}{2} n^{\frac{3}{2}}$ (sharp)
- Thm: Given integers s, t s.t. $s \leq t$, $\exists c$ s.t. $ex(n, K_{s,t}) \leq c \cdot n^{2 - \frac{1}{s}}$
  - sharp when $t \geq (s-1)! + 1$ (i.e. $ex(n, K_{s,t}) \in \Theta(n^{2 - \frac{1}{s}})$)

- **Thm**: For any k : $\exists c$ s.t. $ex(n, C_{2k}) \leq c \cdot n^{1+\frac{1}{k}}$ (i.e. $ex(n, C_{2k}) \in O(n^{1+\frac{1}{k}})$)
  - Wenger's construction: for $k \in \{2,3,5\}$, the bound is tight
- Sidorenko's conjecture: For any bipartite H, and graph G with edge density $p > 0$, there are at least $(p^{e(H)} + o(1)) \cdot n^{v(H)}$ labelled copies of H in G.
  - known for H that are trees, even cycles, complete bipartite graphs.

Spectral Graph Theory:

- spectrum of a graph G: collection of eigenvalues of A_G
  - collection: can have repeats
  - A_G: adj. matrix of G
- Spectral thm: If A is a $n \times n$ real symmetric matrix then there exists $\lambda_1, \dots, \lambda_n$ and mutually orthogonal vectors $\vec{v}_1, \dots, \vec{v}_n$ s.t. $\vec{v}_i$ is an eigenvector of A for eigenvalue λ_i.
- Assume $\lambda_1 \geq \dots \geq \lambda_n$
- $G = K_n$: Spec $= \{n-1, \underbrace{-1, \dots, -1}_{n-1 \text{ times}}\}$
- $G = K_{m,n}$: Spec $= \{\sqrt{mn}, \underbrace{0, \dots, 0}_{m+n-2 \text{ times}}, -\sqrt{mn}\}$
- $G = C_n$: Spec $= \{2\cos(\frac{2\pi j}{n})\}_{j=0,\dots,n-1}$
- If G is d-regular then d is an eigenvalue of A_G
- Circulant matrix: $A_{ij} = A_{0,j-i}$ for all i,j.
  - A_{C_n} is circulant
  - the eigenvalues of a circulant matrix are $\{\sum_{i=0}^{n-1} c_i \omega^i\}_{\omega \text{ is an } n^{th} \text{ root of unity}}$ where $(c_0, \dots, c_{n-1})$ is the first row of the matrix
- $G = P_n$: Spec $= \{2\cos(\frac{\pi j}{n+2})\}_{j=1,\dots,n+1}$
  - P_n: path of n edges & $n+1$ vertices
- Perron-Frobenius thm: (1) If an $n \times n$ matrix A has only nonnegative entries, then:
  - There exists an eigenvalue λ with eigenvector $\vec{v}$
  - λ is nonnegative real
  - λ has largest absolute value amongst all eigenvalues
  - $\vec{v}$ is nonnegative (i.e. $v_i \geq 0$ for all i)

  (2) If in addition A has no $k \times (n-k)$ blocks of zeroes disjoint from the diagonal, then:
  - λ has multiplicity 1
  - $\vec{v}$ is strictly positive (i.e. $v_i > 0$ for all i)



- Applied to graph G:
  - λ_1 has largest absolute value
  - G is connected $\Rightarrow \lambda_1$ has multiplicity 1
- Prop: $\bar{d}(G) \leq \lambda_1(G) \leq \Delta(G)$
  - $\bar{d}$: avg. degree
- Thm: Given a symmetric $n \times n$ matrix A : $\lambda_1(A) = \max_{\vec{x} \in \mathbb{R}^n} \frac{\vec{x}^T \cdot A \cdot \vec{x}}{\vec{x}^T \cdot \vec{x}} = \max_{\vec{x} : \|\vec{x}\|_2 = 1} \vec{x} \cdot A \cdot \vec{x}$
- Prop: For any $S \subseteq V(G)$: $\lambda_1 \geq \bar{d}(G[S])$
  - $G[S]$: S-induced subgraph of G
- Prop: $\sqrt{\Delta(G)} \leq \lambda_1(G) \leq \Delta(G)$
- symmetric $m \times m$ matrix B is a compression of symmetric $n \times n$ matrix A (where $m < n$) : $\exists$ $n \times m$ matrix P s.t. (1) $P^T \cdot P = I_{m \times m}$ (2) $P^T \cdot A \cdot P = B$
- Couchy interlacing thm: If B is a compression of A and Spec(A): $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, Spec$(B)$: $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$ then for all $1 \leq i \leq m$: $\lambda_{i+n-m} \leq \mu_i \leq \lambda_i$

- Min-max thm: If A is an $n \times n$ symmetric matrix with Spec(A): $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, then:

$$\max_{\substack{U \text{ subsp. of } \mathbb{R}^n \\ \dim U = k}} \min_{0 \neq \vec{x} \in U} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \lambda_k = \min_{\substack{U \text{ subsp. of } \mathbb{R}^n \\ \dim U = n+1-k}} \max_{0 \neq \vec{x} \in U} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}$$

- Linear algebra:
  - Trace := sum of elements on the main diagonal = sum of eigenvalues
  - Determinant := $\det(A)$ = product of eigenvalues
- Corollary from CIT: $\alpha(G) \leq \min\{n_{\geq 0}(A_G), n_{\leq 0}(A_G)\}$
  (Inertia bound)
  number of non-negative eigenvalues of A_G
- Hoffman bound: If G is regular then: $\alpha(G) \leq \frac{-\lambda_n}{\lambda_1 - \lambda_n} \cdot n$
  - Cor: If G is regular then: $\chi(G) \geq \frac{\lambda_1 - \lambda_n}{-\lambda_n}$
- Stronger thm: $\chi(G) \geq \frac{\lambda_1 - \lambda_n}{-\lambda_n}$ (whether or not G is regular)
- EKR thm: If $n \geq 2k$, then: the largest family $\mathcal{F}$ of k-subsets on $[n]$ for which $\forall A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$ has $\binom{n-1}{k-1}$ sets.
  (consider all the k-subsets containing a fixed element)
- Thm: $\chi(KG(n,k)) = n - 2k + 2$
  Kneser graph: vertices are the k-subsets of $[n]$, and two vertices are adjacent if the two sets are disjoint.
- Wilf thm: $\chi(G) \leq \lfloor \lambda_1 \rfloor + 1$
- Thm: If G is connected then: $\lambda_n = -\lambda_1 \iff G$ is bipartite
- Friendship theorem: If any two people have exactly one mutual friend then there is one person that is a friend of everyone else.

- Spectrum of the complement: If $\mathrm{Spec}(G) = \{\lambda_1, \dots, \lambda_n\}$
  and $\mathrm{Spec}(\overline{G}) = \{\mu_1, \dots, \mu_n\}$
  then $\lambda_1 + \mu_1 = n-1$ and for $i \in \{2, \dots, n\}$: $\lambda_i + \mu_{n+2-i} = -1$